REAL ANALYSIS
TOPIC 36 - INTEGRATION

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ABSTRACT. We review the definition of the Riemann integral, and introduce the Darboux integral. We state the relationship between them. We define the Lebesgue integral in multiple ways, and discuss their relations.

1. The Riemann Integral

Definition 1. Let $a, b \in \mathbb{R}$ with $a < b$.

A *partition* of the closed interval $[a, b]$ is a finite set

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

with the property that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$. We view P as indicating a way of breaking the interval $[a, b]$ into n subintervals. The width of the i^{th} subinterval is $\Delta x_i = x_i - x_{i-1}$, for $i = 1, \dots, n$.

The *norm* of the partition P is

$$||P|| = \max\{\Delta x_i \mid i = 1, \dots, n\}.$$

A *choice set* for P is a finite set

$$C = \{x_1^*, x_2^*, \dots, x_n^*\}$$

such that $x_i^* \in [x_{i-1}, x_i]$, for $i = 1, \dots, n$. Note that this implies

$$x_1^* < x_2^* < \cdots < x_n^*.$$

Let $f : [a, b] \to \mathbb{R}$. The *Riemann sum* associated to a partition P and a choice set C for P is

$$R(f, P, C) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

We say that f is *Riemann integrable with integral I* if there exists a real number $I \in \mathbb{R}$ such that, for every positive real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every partition P and choice set C of P,

$$\|P\| < \delta \quad \Rightarrow \quad |R(f, P, C) - I| < \epsilon.$$

If f is Riemann integrable with integral I, we write

$$\int_a^b f(x)\, dx.$$

This is read, "the integral from a to b of $f(x)\, dx$".

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2. Darboux Integral

2.1. Partitions.

Definition 2. Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* of the closed interval $[a, b]$ is a finite set

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

with the property that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

This notion of partition is the same as it was for the Riemann integral. We write a partition P as a set, but it is in fact an ordered set, and by convention, the order is dictated by the indices of the points in the set. We view P as indicating a way of breaking the interval $[a, b]$ into n closed subintervals, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

Definition 3. Let $a, b \in \mathbb{R}$ with $a < b$. Let P be a partition of $[a, b]$. A *refinement* of P is a partition Q of $[a, b]$ such that $P \subset Q$.

Proposition 1. *Any two partitions of* $[a, b]$ *have a common refinement.*

Proof. Let P_1 and P_2 be partitions of $[a, b]$. Then $Q = P_1 \cup P_2$ is a refinement of P_1 and of P_2. □

2.2. Darboux Sums.

Definition 4. Let f be a bounded function defined on a closed interval $[a, b]$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. Set

$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\},$ and $\Delta x_k = x_k - x_{k-1}\},$

for $i = 1, \ldots, n$.

The *lower Darboux sum* of f over P is

$$\underline{S}(f, P) = \sum_{k=1}^{n} m_k \, \Delta x_k.$$

The *upper Darboux sum* of f for P is

$$\overline{S}(f, P) = \sum_{k=1}^{n} M_k \, \Delta x_k.$$

The key difference between the definitions of the Riemann and Darboux integral lies in Darboux use of the infimum and supremum, as above, instead of the "choice set" in the Riemann integral, which can be slightly unwieldly.

It is clear from the definition that $\underline{S}(f, P) \leq \overline{S}(f, P)$.

Proposition 2. *Let f be a bounded function defined on a closed interval $[a, b]$. Let P be a partition of $[a, b]$, and let Q be a refinement of P. Then*

$$\underline{S}(f, P) \leq \underline{S}(f, Q) \leq \overline{S}(f, Q) \leq \overline{S}(f, P).$$

Proof. We discuss the last two inequalities, the first one being similar to the last.

Consider the middle inequality. It states that $\sum_{k=1}^{n} m_k \, \Delta x_k \leq \sum_{k=1}^{n} M_k \, \Delta x_k$. But this is clear, since $m_k = \inf A \leq \sup A = M_k$ for $A = f([x_{k-1}, x_k])$.

Consider the last inequality. If $P = Q$, we have equality here. Otherwise, Q contains at least one more point than P; let us suppose that Q contains exactly one more point than P. This point is in one of the subintervals determined by P, say $y \in Q$ and $x_{k-1} < y < x_k$. Then

$$\begin{aligned}
\overline{S}(f, P) - \overline{S}(f, Q) &= (\sup f([x_{k-1}, x_k]))(x_k - x_{k-1}) \\
&\quad - (\sup f([x_{k-1}, y]))(y - x_{k-1}) + (\sup f([y, x_k]))(x_k - y) \\
&= (\sup f([x_{k-1}, x_k]) - \sup(f([x_{k-1}, y])))(y - x_{k-1}) \\
&\quad + (\sup f([x_{k-1}, x_k]) - (\sup f([y, x_k])))(x_k - y) \\
&\geq 0,
\end{aligned}$$

since $B \subset A$ implies $\sup A \geq \sup B$. Since the inequality holds if we add one point to P, it will continue to hold as we add more points. □

2.3. **Darboux Integral.**

Definition 5. Let f be a bounded function defined on a closed interval $[a, b]$.

The *lower Darboux integral* of f on $[a, b]$ is

$$\underline{\int_a^b} f = \sup\{\underline{S}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

The *upper Darboux integral* of f on $[a, b]$ is

$$\overline{\int_a^b} f = \inf\{\overline{S}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Many books call these the lower and upper Riemann integral.

Proposition 3. *Let f be a bounded function defined on a closed interval $[a, b]$. Let P be a partition of $[a, b]$. Let $m = \inf\{f(x) \mid x \in [a, b]$ and $M = \sup\{f(x) \mid x \in [a, b]\}$. Then*

$$m(b - a) \leq \underline{S}(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \overline{S}(f, P) \leq M(b - a).$$

Proof. We discuss the last three inequalities, as the first two are analogous to the last two.

The last inequality is obtained by setting $Q = \{a, b\}$, so that P is a refinement of Q. Then

$$\overline{S}(f, P) \leq \overline{S}(f, Q) = M(b - a).$$

That $\overline{\int_a^b} f \, dx \leq \overline{S}(f, P)$ follows from the fact that $\overline{\int_a^b} f$ is the supremum of a set which contains $\overline{S}(f, P)$.

That $\underline{\int_a^b} f \, dx \leq \overline{\int_a^b} f$ follows from the fact that if $a \leq b$ for every $a \in A$ and $b \in B$, then $\sup A \leq \inf B$. □

Definition 6. Let f be a bounded function defined on a closed interval $[a,b]$. We say that f is *Darboux integrable* on $[a,b]$ if

$$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx.$$

In this case, the common value is called the *Darboux integral* of f on $[a,b]$, and is denoted

$$\int_a^b f\,dx.$$

Proposition 4. *Show that f is Darboux integrable on $[a,b]$ if and only if, for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$|\overline{S}(f,P) - \underline{S}(f,P)| < \epsilon.$$

Proposition 5. *Let $f : [a,b] \to \mathbb{R}$. Then f is Riemann integrable if and only if f is Darboux integrable, and in the case these integrals exist, they are equal.*

Proposition 6. *Let $f : [a,b] \to \mathbb{R}$. Then f is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.*

Example 1. Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational ;} \\ 1 & \text{if } x \text{ is rational .} \end{cases}$$

Every subinterval of every partition contains an open interval which contains both rational and irrational numbers. Thus $m_k = 0$ and $M_k = 1$ for all subintervals, whence $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$. Thus f is not Riemann integrable.

Example 2. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Then f is Riemann integrable, and in fact, $\int_a^b f = 0$.

3. Lebesgue Integral

The goal of modifying the definition of integration is to increase the number of integrable functions without changing the value of the integrals. The Lebesgue integral was created with this in mind. The tool that is used to do this is measure; in essence, we replace in the definition the length of intervals with the measures of measurable sets.

The Lebesgue integral has been described as "vertical integration"; that is, breaking the range up into subintervals instead of breaking up the domain. We will describe the approach from Saxena-Shah first, as it is a direct generalization of the Darboux integral.

3.1. **Partitions.** We start by generalizing the Riemann integral's concept of "partition".

Definition 7. Let $D \subset \mathbb{R}$ be a measurable set. A *partition* of D is a finite collection $\mathcal{P} = \{D_1, \ldots, D_n\}$ of measurable subsets of D such that

- $\cup_{k=1}^{n} D_k = D$, and
- $D_i \cap D_j = \varnothing$ unless $i = j$.

Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* of the closed interval $[a, b]$ is a finite set

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

with the property that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

3.2. **Lebesgue Sums.**

Definition 8. Let f be a bounded function defined on a measurable set D. Let $\mathcal{P} = \{D_1, \ldots, D_n\}$ be a partition of D. Set

$$m_k = \inf\{f(x) \mid x \in D_k\} \text{ and } M_k = \sup\{f(x) \mid x \in D_k\},$$

for $k = 1, \ldots, n$. Set

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k], \text{ and } \Delta x_k = x_k - x_{k-1}\},$$

for $k = 1, \ldots, n$.

The *lower Lebesgue sum* of f over $\mathcal{P}$ is

$$\underline{S}(f, \mathcal{P}) = \sum_{k=1}^{n} m_k \, \mu(D_k).$$

The *upper Lebesgue sum* of f for $\mathcal{P}$ is

$$\overline{S}(f, \mathcal{P}) = \sum_{k=1}^{n} M_k \, \mu(D_k).$$

It is clear from the definition that $\underline{S}(f, P) \leq \overline{S}(f, P)$.

3.3. **Lebesgue Integral.**

Definition 9. Let f be a bounded function defined on a measurable set D.

The *lower Lebesgue integral* of f on D is

$$\underline{\int_D} f\,d\mu = \sup\{\underline{S}(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } D\}.$$

The *upper Lebesgue integral* of f on $[a,b]$ is

$$\overline{\int_D} f\,d\mu = \inf\{\overline{S}(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } D\}.$$

Definition 10. Let f be a bounded function defined on a measurable set D. We say that f is *Lebesgue integrable* on D if

$$\underline{\int_D} f\,d\mu = \overline{\int_D} f\,d\mu.$$

In this case, the common value is called the *Lebesgue integral* of f on D, and is denoted

$$\int_D f\,d\mu.$$

4. Vertical Integral

We discuss a manner of creating a integral by breaking up the range, as opposed to the domain, into subintervals.

Definition 11. Let f be a bounded function defined on a measurable set D. Let $[c,d] \subset \mathbb{R}$ such that $\operatorname{range} f \subset [c,d]$. For convenience of notation, we assume $d > \sup \operatorname{range} f$.

Let P be a partition of $[c,d]$ in the sense of the Riemann integral; that is, $P = \{y_0, y_1, \dots, y_n\}$, where

$$c \leq y_0 \leq y_1 \leq \cdots \leq y_n = d.$$

Let C be a choice set of the partition P, so that $C = \{y_1^*, \dots, y_n^*\}$ where $c_k^* \in [y_{k-1}, y_k\}$.

Let $D_k = \{x \in D \mid y_{k-1} \leq f(x) < y_l\}$. That is, D_k is the preimage of the k^{th} subinterval of the range. We assume that D_k is measurable.

The "vertical sum" of f with respect to P and C is

$$S(f,P,C) = \sum_{k=1}^{n} y_k^* \mu(D_k).$$

We say that f is *Vertically integrable with integral* I if there exists a real number $I \in \mathbb{R}$ such that, for every positive real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every partition P and choice set C of P,

$$\|P\| < \delta \quad \Rightarrow \quad |S(f,P,C) - I| < \epsilon.$$

Here, I is the value of the integral, if it exists.

Notice that we could also define this in a manner similar to the Darboux integral, using the sup and inf.

5. Step Integral

Many books use the following definition for the Lebesgue integral. We will use a different name, just to distinguish between the two. This definition is more complex, but can be used in proofs to develop the theory more completely.

5.1. **Characteristic and Simple Functions.**

Definition 12. Let $A \subset D \subset \mathbb{R}$. The *characteristic function* of A in D is

$$\chi_A : D \to \mathbb{R} \quad \text{given by} \quad f(x) = \begin{cases} 1 & \text{if } x \in A\ ; \\ 0 & \text{if } x \notin A\ . \end{cases}$$

Definition 13. Let $D \subset \mathbb{R}$ be a measurable with finite measure. A *simple function* on D is a function $\phi : D \to \mathbb{R}$ whose range is finite, such that the preimage of each value in the range is measurable.

Note that if ϕ and ψ are simple functions on D, and $a, b \in \mathbb{R}$, then $a\phi + b\psi$ is a simple function on D.

Let ϕ be a simple functions whose distinct values are $y_1, \dots, y_n$. Let $D_k = \phi^{-1}(y_k)$. Then ϕ by be expressed as

$$\phi = \sum_{k=1}^{n} y_k \xi(D_k).$$

Definition 14. Let $\phi : D \to \mathbb{R}$ be a simple function with distinct values $y_1, \dots, y_n$. Let $D_k = \phi^{-1}(y_k)$. The Step integral of ϕ is

$$\int_D \phi \, d\mu = \sum_{k=1}^{n} y_k \mu(D_k).$$

Definition 15. Let $f : D \to \mathbb{R}$ be a bounded function on a measurable set D of finite measure.

The *lower Step integral* of f on D is

$$\underline{\int_D} f \, d\mu = \sup_{\phi \le f} \int \phi \, d\mu,$$

where $\sup_{\phi \le f}$ is the supremum on the collection of all simple functions ϕ on D such that $\phi \le f$ on D.

The *upper Step integral* of f on D is

$$\overline{\int_D} f \, d\mu = \inf_{\psi \ge f} \int \psi \, d\mu,$$

where $\inf_{f \le \psi}$ is the infimum on the collection of all simple functions ψ on D such that $\psi \ge f$ on D.

Observation 1. Let $f : D \to \mathbb{R}$ be a bounded function on a measurable set D of finite measure. Suppose $f(x) \in [m, M]$ for all $x \in D$, where $m, M \in \mathbb{R}$. Then

$$m\mu(D) \le \underline{\int_D} f \, d\mu \le \overline{\int_D} f \, d\mu \le M\mu(D).$$

Definition 16. Let $f : D \to \mathbb{R}$ be a bounded function on a measurable set D of finite measure. We say that f is *Stepwise integrable* on D if

$$\underline{\int_D} f \, d\mu = \overline{\int_D} f \, d\mu.$$

In this case, the common value is called the *Step integral* of f, given as

$$\int_D f \, d\mu = \underline{\int_D} f \, d\mu.$$

6. So now what?

The question becomes, are any of these definitions different? The following is relatively clear.

Observation 2. If f is Lebesgue integrable on D, then f is Darboux integrable on D, and so f is Riemann integrable on D.

Reason. Every partition in the sense of Darboux induces a partition in the sense of Lebesgue. □

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